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Lyapunov density for coupled systems

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We prove a necessary and sufficient condition for the existence of Lyapunov density for a system of coupled autonomous ordinary differential equations. In particular, we characterize the kinds of couplings that preserve almost everywhere uniform stability of the origin provided the isolated systems have an almost everywhere uniformly stable equilibrium point at the origin.

Keywords: almost everywhere uniform stability; advection equation; density function

AMS Subject Classifications: Primary: 93D05; Secondary: 93B07

1. Introduction

The concept of almost everywhere (a.e.) stability of an equilibrium point for a nonlinear autonomous differential equation was first introduced by Prof. Rantzer in [1,2]. If the equilibrium point satisfies the condition that almost every (in the sense of Lebesgue measure) initial condition reaches the equilibrium point asymptotically in time, then the equilibrium point is classified as a.e. stable. The main result was that a.e. stability of an equilibrium point is equivalent to the existence of a density function \( \rho(x) \in C^1(\mathbb{R} \setminus \{0\}) \), assuming without loss of generality that equilibrium point is \( x = 0 \). Several papers have appeared thereafter talking about the computation of the density function using polynomial and other methods.[3,4]

The Perron–Frobenius (P–F) transfer operator-based framework was developed for a.e. and almost everywhere uniform (a.e.u.) stability analysis of general attractor sets in discrete-time systems.[5,6] The transfer operator-based framework was used to strengthen the concept of a.e. stability to the notion of a.e.u. stability of an attractor set \( \Lambda \) for continuous time systems in [7,8]. The strengthening is very much akin to the relationship between asymptotic stability and exponential stability of \( \Lambda \),[9] in the sense that a.e.u. stability requires that almost every initial condition reaches \( \Lambda \) fast enough to allow for an integrability condition to be true (see Section 2).

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In [9], it was shown that if a system can be modeled as an interconnection of lower order subsystems each of which is stable and has a positive definite Lyapunov function, then provided the interconnections satisfied a certain mathematical criterion, then using the Lyapunov functions for the lower order systems, a Lyapunov function can be constructed for the system with the interconnections, thereby guaranteeing stability for the interconnected system. We show that a similar argument can be used to derive a necessary and sufficient condition for a.e.u. stability of the origin for a coupled system.

The paper is organized as follows: In Section 2, we recall several important results that were proved in [8]. In Section 3, we state and prove a result that characterizes the kinds of vanishing interconnections that preserve a.e.u. stability of the origin for systems possessing a.e.u. stable equilibrium points at the origin. We discuss a few examples in Section 4, and conclude with some remarks in Section 5.

2. Preliminary results

We consider the following autonomous differential equation:

\[ \dot{x} = f(x), \quad x \in X, \] (1)

where the vector field \( f \) is assumed to be infinitely smooth and \( X \subset \mathbb{R}^n \) is a compact phase space that is positively invariant for (1). We use the notation \( \phi_t(x) \) to denote the solution or flow map of (1) at time \( t \), having started from the initial condition \( x \). Equation (1) can be used to study the evolution of a single trajectory. The evolution of ensembles of trajectories or densities in phase space can be studied using a linear operator called the P–F operator \( \mathbb{P}_t : L^1(X) \to L^1(X) \) which satisfies the following conservation property:

\[ \int_A \mathbb{P}_t \rho(x) dx = \int_{\phi_t(A)} \rho(x) dx = \int_A \rho(\phi_{-t}(x)) \left| \frac{\partial \phi_{-t}(x)}{\partial x} \right| dx \] (2)

for every measurable set \( A \subset X \). Hence, the following identity is true

\[ \mathbb{P}_t \rho(x) = \rho(\phi_{-t}(x)) \left| \frac{\partial \phi_{-t}(x)}{\partial x} \right|, \] (3)

where \( |\frac{\partial \phi_{-t}(x)}{\partial x}| \) is the determinant of the Jacobian of the flow map \( \phi_{-t} \). Furthermore, the P–F operator introduced above is the semigroup corresponding to the operator \( \mathbb{A} \rho = -\nabla \cdot (\rho f) \). In other words, \( \mathbb{P}_t \rho_0 = e^{\mathbb{A}t} \rho_0(x) \) describes the evolution of densities \( \rho \) via the advection equation

\[ \frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho f) =: \mathbb{A} \rho; \quad \rho(x, 0) = \rho_0(x). \] (4)

If \( (X, \mathcal{B}, \mu) \) is a measure space and \( \mathbb{P}_t \) is the P–F operator corresponding to the dynamical system (1), then \( \mathbb{P}_t \) satisfies the following properties [10]:

1. \( \mathbb{P}_t(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 \mathbb{P}_t f_1 + \alpha_2 \mathbb{P}_t f_2 \) for all \( f_1, f_2 \in L^1(X) \) and \( \alpha_1, \alpha_2 \in \mathbb{R} \).
2. \( \mathbb{P}_t f \geq 0 \) if \( f \geq 0 \).
3. \( \int_X \mathbb{P}_t f(x) \mu(dx) = \int_X f(x) \mu(dx) \).

We recall the definitions of an \( \omega \)-limit set and an attractor set below:
Definition 2.1 (ω-limit set) A point $x_0 \in X$ is said to be an ω-limit point for a point $x \in X$ if there exists a sequence of time instants $t_k \to \infty$ such that $\phi_{t_k}(x) \to x_0$ as $k \to \infty$. The set of all ω-limit points $\omega(x)$ for $x$ is called the ω-limit set $\omega(x)$.

The definition of an attractor set is as follows:

Definition 2.2 (Attractor set) A closed set $\Lambda \subset X$ is said to be an invariant set for (1) if for any $x \in \Lambda$, $\phi_t(x) \in \Lambda$ for all $t \in \mathbb{R}$. An invariant set $\Lambda$ is said to be an attractor set if there exists a neighborhood $V \supset \Lambda$ such that $\omega(x) \subset \Lambda$ for all $x \in V$, and the neighborhood $V$ is forward invariant, i.e. $\phi_t(x) \in V$ for all $t \geq 0$ and for all $x \in V$.

We will use $\Lambda$ to denote an attractor set i.e. $\Lambda$ itself is an invariant set and there is a neighborhood $V$ that is assumed to be present for $\Lambda$ satisfying Definition (2.2). Furthermore, we will assume without loss of generality that there exists a $\delta > 0$ and a ball $B_\delta$ such that

$$\Lambda \subset B_\delta \subset V$$

in all the definitions that follow. We define a new semigroup corresponding to the restriction of the flow $\phi_t : X \setminus \Lambda \to X \setminus \Lambda$ as follows:

$$\mathbb{P}_t^1 \rho(x) := \rho \left( \phi_{-t}(x) \right) \frac{\partial \phi_{-t}(x)}{\partial x},$$

where $\rho(x)$ is supported on the set $X \setminus \Lambda$.

Since $\Lambda$ is assumed to be invariant for the dynamics defined by (1), so is $X \setminus \Lambda$ and hence (6) defines a semigroup on $X \setminus \Lambda$.

We have the following:

$$\mathbb{P}_t^1 = \Sigma \mathbb{P}_t : L^1(X \setminus \Lambda) \to L^1(X \setminus \Lambda),$$

where $\Sigma : L^1(X) \to L^1(X \setminus \Lambda)$ is the projection operator defined by

$$(\Sigma \rho)(x) = \chi_{X \setminus \Lambda}(x) \rho(x),$$

where $\chi_{X \setminus \Lambda}(x)$ is the characteristic function of the set $X \setminus \Lambda$. Let $\mathbb{A}^1$ be the infinitesimal generator corresponding to the semigroup of the restriction $\mathbb{P}_t^1$. The domain of the generators $\mathbb{A}$ and $\mathbb{A}^1$ are denoted by $\mathcal{D}(\mathbb{A})$ and $\mathcal{D}(\mathbb{A}^1)$, respectively, and are defined below:

$$\mathcal{D}(\mathbb{A}) = \left\{ \rho \in W^{1,1}(X) : \rho|_{\Gamma_i} = 0 \right\},$$

$$\mathcal{D}(\mathbb{A}^1) = \left\{ \rho \in W^{1,1}(X \setminus \Lambda) : \rho|_{\Gamma_i} = 0 \right\},$$

where $W^{1,1}(X)$ is the Sobolev space of distributions in $L^1(X)$ whose first weak derivative is also a distribution in $L^1(X)$, and the inflow portion of the boundary of $X$ (if it exists) is denoted by $\Gamma_i$ and is given as follows:

$$\Gamma_i = \left\{ x \in \partial X : \tilde{n}(x) \cdot \eta(x) < 0 \right\},$$

where $\tilde{n}(x)$ is the unit outward normal at the boundary point $x$ and $\partial X$ denotes the set of all boundary points of $X$. For sets $X$ that do not have a boundary, the homogeneous boundary conditions can be omitted from the domain definitions in (7).
We review some important results proved in [8] related to a.e.u. stability of an attractor set Λ.

The main reason behind choosing to study the stability of an attractor set rather than an invariant set is because it allows us to impose boundary conditions and compute the Lyapunov density as the solution of a linear partial differential equation.

**Definition 2.3** (Almost everywhere stability) Let

\[ A_t = \{ x \in X \setminus \Lambda : \phi_t(x) \in A \} \]

An attractor set Λ is said to be almost everywhere stable with respect to a finite measure \( m \) on \( X \setminus \Lambda \) if \( m\{x \in X \setminus \Lambda : \omega(x) \not\subset \Lambda \} = 0 \).

**Definition 2.4** (Almost everywhere uniform stability) Let \( \Lambda \) be an attractor set and \( B_\delta \) be as in (5). Then, \( \Lambda \subset B_\delta \) for the differential Equation (1) is said to be almost everywhere uniformly stable with respect to a finite measure \( m \) on \( X \setminus \Lambda \) if for any given \( \delta > 0 \) and \( \epsilon > 0 \), there exists a \( T(\epsilon) > 0 \) such that

\[
\int_T^\infty m(A_t) dt < \epsilon
\]

for all measurable \( A \subset X \setminus B_\delta \), where \( B_\delta \) is a \( \delta \) neighborhood of \( \Lambda \).

**Remark 1** The set \( B_\delta \) in Definition (2.4) essentially allows us to talk about evolution of densities that are integrable in steady state and avoids singularities near the invariant set \( \Lambda \) due to accumulation of mass. Hence, we assume that the initial density will be supported on \( X \setminus \Lambda \) and focus on this set to define almost everywhere uniform stability of the attractor set \( \Lambda \).

Roughly speaking, this means that the total mass outside of the \( \delta \) neighborhood of \( \Lambda \) goes to zero asymptotically in time. In addition, this also imposes a certain restriction on the speed of movement of mass from \( X \setminus B_\delta \) to \( B_\delta \).

**Definition 2.5** (Lyapunov density) Let \( \Lambda \) be an attractor set and \( \delta > 0 \) be such that \( \Lambda \subset B_\delta \) as in (5). Let \( m \) be a measure that is assumed to be absolutely continuous with respect to the Lebesgue measure on \( X \setminus \Lambda \) with density \( \rho_0(x) \in L^1(X \setminus B_\delta) \). A nonnegative function \( \rho(x) \in \mathcal{D}(\mathcal{H}^1) \) is said to be a Lyapunov density with respect to density \( \rho_0 \) if it satisfies the following inequality

\[
\mathcal{H}^1 \rho(x) \leq -\rho_0(x).
\]

**Remark 2** Inequality (10) is the same as

\[
\nabla \cdot (f \rho) \geq \rho_0(x).
\]

We recall three important theorems regarding a.e.u. stability of an attractor set \( \Lambda \) from [8].
**Theorem 2.6** Let $X \subset \mathbb{R}^n$ be compact and $\Gamma_1$ denote the inflow part of the boundary $\partial X$ which is assumed to be $C^2$. Then, the attractor set $\Lambda$ is almost everywhere uniformly stable with respect to the measure $m$ with density $0 \leq \rho_0 \in L^1(X \setminus \Lambda)$ if and only if there exists a Lyapunov density with respect to density $\rho_0$.

**Theorem 2.7** Let $X \subset \mathbb{R}^n$ be compact. Furthermore, let us assume that the attractor set $\Lambda$ is almost everywhere uniformly stable with respect to the measure $m$ with density $0 \leq \rho_0 \in L^1(X \setminus \Lambda)$. Then, the Lyapunov density with respect to density $\rho_0$ can be computed by solving the following linear partial differential equation:

\[ A^1_1 \rho(x) = -\rho_0(x), \quad (12) \]

with the following homogeneous Dirichlet boundary conditions

\[ \rho|_{\Gamma_1} = 0. \quad (13) \]

**Theorem 2.8** The attractor set $\Lambda \subset X$ for the system of differential Equation (1) is almost everywhere uniformly stable with respect to Lebesgue measure on $X \setminus \Lambda$ if and only if it is almost everywhere uniformly stable with respect to every finite measure $m$ with density $0 \leq \rho_0 \in L^1(X \setminus \Lambda)$.

**Remark 3** It was shown in [8] that the Lyapunov density has the representation formula

\[ \rho(x) = \int_0^\infty P^1_t \rho_0(x) dt, \]

where $P^1_t : L^1(X \setminus \Lambda) \to L^1(X \setminus \Lambda)$ given by $P^1_t \rho = \rho(\phi_{-t}(x)) \text{det}(\frac{d(\phi_{-t}(x))}{dx})$ is the P–F semigroup operator whose generator is the operator $A^1_1$ defined above. It is clear from this formula that if trajectories move very slowly, then $\rho(x)$ might develop a singularity thereby violating the $L^1$ integrability requirement in space.

Theorem (2.6) characterizes a.e.u. stability in terms of the existence of a Lyapunov density as a certificate, Theorem (2.7) allows us to compute the Lyapunov density by solving a linear PDE, and Theorem (2.8) allows us to choose $\rho_0(x) = \chi_{X \setminus B_\delta}$ in (12) while computing the density.

### 3. Main result

We introduce some notation to set things up first. We consider the following coupled system:

\[ \dot{x}_i = f_i(x_i) + g_i(x), \quad i = 1, 2, 3, \ldots, m, \quad (14) \]

where $x_i \in \mathbb{R}^{n_i}, \sum_{i=1}^m n_i = n$, and $x = (x_1^T, x_2^T, \ldots, x_m^T)^T$. We assume that the following are true of the coupled system (14) above:

1. $f_i(x_i)$ and $g_i(x)$ are smooth enough to ensure existence and uniqueness of solutions for the isolated (i.e. $g_i(x) = 0$ in (14)) systems on respective compact sets $X_i \subset \mathbb{R}^{n_i}$ and the coupled system (14) on a compact set $X = \times_{i=1}^m X_i \subset \mathbb{R}^n$.
2. $f_i(0) = 0$ and $g_i(0) = 0 \forall i = 1, 2, \ldots, m$ so that $x = 0$ is an equilibrium point of the coupled system (14).
We also have that $a$-e.u. stable equilibrium point $x_i = 0 \in \mathbb{R}^{n_i}$. Hence, by Theorems (2.6–2.8), we have that each of the subsystems in (15) possesses a Lyapunov density $\rho_i(x_i) \in L^1(X_i \setminus \{0\})$ satisfying $\nabla \cdot (f_i \rho_i) = 1$ (or $\chi_{X_i \setminus \{0\}}(x_i)$).

Theorem 3.1 Let the coupled system (14) satisfy the assumptions (1–4) above. Then $x = 0 \in \mathbb{R}^n$ is an a.e.u. stable equilibrium point for (14) if and only if the interconnections $g_i(x)$ satisfy the following:

$$\exists \epsilon > 0 \ni \nabla_x \cdot \left( g(x) \prod_{i=1}^{m} \rho_i(x_i) \right) + \sum_{i=1}^{m} \prod_{j=1, j \neq i}^{m} \rho_j(x_j) \geq \epsilon$$

(a.e. $x \in X$, where $g = (g_1^T, g_2^T, \ldots, g_m^T)^T$.

Proof (⇐) Since $x_i = 0 \in \mathbb{R}^{n_i}$ is a.e.u. stable for $\dot{x}_i = f_i(x_i)$, we can first solve $\nabla x_i \cdot (f_i \rho_i) = 1; \rho_i|_{\Gamma_i} = 0$ for $\rho_i$ by Theorem (2.7) (where we have chosen $\rho_0(x) = \chi_{X_i \setminus \{0\}}(x)$ and $\Gamma_i = \partial X_i$). Let $F = ((f_1 + g_1)^T, (f_2 + g_2)^T, \ldots, (f_m + g_m)^T)^T$ and $\rho = \prod_{i=1}^{m} \rho_i$. Then the proof of this part boils down to showing that there exists a Lyapunov density for $\dot{x} = F(x)$ thereby implying a.e.u. stability of $x = 0 \in \mathbb{R}^n \subset X = \times_{i=1}^{m} X_i$. We will show that $\rho = \prod_{i=1}^{m} \rho_i$ will satisfy just that. We have the following:

$$\nabla_x \cdot (F \rho) = \sum_{i=1}^{m} \left( \prod_{j=1, j \neq i}^{m} \rho_j(x_j) \right) \nabla x_i \cdot (f_i(x_i) \rho_i(x_i)) + \nabla_x \cdot (g(x) \rho(x))$$

$$= \sum_{i=1}^{m} \prod_{j=1, j \neq i}^{m} \rho_j(x_j) + \nabla_x \cdot (g(x) \rho(x)) \geq \epsilon > 0.$$ 

Furthermore, we have that $\rho|_{\Gamma_i} = 0$ by definition. Now we choose $\tilde{\rho} = \frac{\rho}{\epsilon}$. Then we get

$$\nabla \cdot ((f + g) \tilde{\rho}) = \frac{1}{\epsilon} \nabla \cdot ((f + g) \rho) \geq \frac{\epsilon}{\epsilon} = 1.$$ 

We also have that $\tilde{\rho}|_{\Gamma_i} = 0$ (where $\Gamma = \times_{i=1}^{m} \Gamma_i$) since $\tilde{\rho}$ is just a scalar multiple of $\rho$. Hence, we have found a solution to the following problem $\nabla \cdot (F \rho) \geq 1; \rho|_{\Gamma_i} = 0$. This means that $x = 0 \in \mathbb{R}^n$ is a.e.u. stable for $\dot{x} = F(x)$, by Theorem (2.6).

(⇒) We prove the other direction by contradiction. Assume that $x_i = 0 \in \mathbb{R}^{n_i}$ is a.e.u. stable for $\dot{x}_i = f_i(x)$ and $x = 0 \in \mathbb{R}^n$ for $\dot{x} = F(x)$, and $f_i(x), g_i(x)$ satisfy
Assume that Theorems (2.6–2.8), there exists a Lyapunov density $0 \leq \rho_i(x_i) \in L^1(X_i \setminus \{0\})$ that satisfies the following PDE on $X_i$:

$$\nabla \cdot (f_i \rho_i) = 1; \quad \rho_i|_{\Gamma_i} = 0. \quad (17)$$

Assume that $g$ satisfies $\nabla \cdot (g\rho) + \sum_{i=1}^n \prod_{j=1}^m \rho_j \leq 0$, on a set $A \subset X$ of positive Lebesgue measure (here once again we use the density. We define $\rho = \rho \in \mathbb{R}^n$ of the coupled system $\dot{x} = F(x)$ is contradicted by showing that $\rho(x) \leq 0$ on $A$.

We have the following calculation:

$$\nabla \cdot (F\rho) = \sum_{i=1}^m \prod_{j=1,j\neq i}^m \rho_j(x_j) + \nabla_x \cdot (g(x)\rho(x)) \leq 0 \forall x \in A, \quad (18)$$

but $\rho > 0$ on $A$. We show that this cannot happen due to the positivity property enjoyed by the density. We define $\rho_0 := \nabla \cdot (F\rho)$. Then we have that $\rho_0 \leq 0$ on $A$ and $\rho$ is the solution of

$$\nabla \cdot (F\rho) = \rho_0 < 0 \text{ on } A. \quad (19)$$

From the assumption that $x = 0 \in \mathbb{R}^n$ is a.e. stable for $\dot{x} = F(x)$, we have from Theorem (2.6) and Remark (3) that the solution of (19) is given by the formula $\rho(x) = \int_0^\infty \hat{\rho}_t^1 \rho_0(x) dt \in L^1(X \setminus \{0\})$, where $\hat{\rho}_t^1$ is the P–F operator corresponding to the ODE $\dot{x} = F(x)$ restricted to $X \setminus \{0\}$. Since $\rho_0(x) \leq 0$, by the positivity property of $\hat{\rho}_t^1$, we have that $\rho(x) \leq 0$ on $A$. This is a contradiction. □

Remark 1 Condition (16) has to be understood in the weak sense (i.e. in the sense of distributions) if the Lyapunov densities $\rho_i$ are not differentiable. Since Lyapunov density only belongs to the $L^1$ class, this might be a possibility.

Next, we specialize Theorem (3.1) above to the case where the interconnection is linear:

**Corollary 3.2 (Linear coupling of scalar systems).** Let the coupled system (14) satisfy the assumptions (1–4) above. Also, let $n_i = 1 \forall i = 1, \ldots, m$, $\rho(x) = \prod_{i=1}^n \rho_i(x_i)$, $g(x) = (g_1(x)^T, g_2(x)^T, \ldots, g_m(x)^T)^T = Lx$ where $L \in \mathbb{R}^{m \times m}$, and $\rho$ is a column vector of functions defined as follows

$$(\hat{\rho}(x))_j = \left( \prod_{i=1,i\neq j}^m \rho_i(x_i) \right) \rho_j'(x_j)$$

where $'$ denotes differentiation with respect to the corresponding single variable. Then $x = 0 \in \mathbb{R}^n$ is an a.e. stable equilibrium point for (14) if and only if the matrix $L$ satisfies the following:

$$\exists \epsilon > 0 \exists x^T L^T \hat{\rho}(x) + Tr(L) \cdot \rho(x) + \sum_{i=1}^m \prod_{j=1, j\neq i}^m \rho_j(x_j) \geq \epsilon \text{ a.e. } x \in X. \quad (20)$$
The proof follows by replacing $g(x) = (g_1(x)^T, g_2(x)^T, \ldots, g_m(x)^T)^T = Lx$ in Theorem (3.1).

We next specialize for the case of a cascade of scalar systems with linear coupling as shown in Figure 1 below:

**Corollary 3.3 (Linear cascade coupling of scalar systems)** Let the coupled system (14) satisfy the assumptions (1–4) above. Also, let $n_i = 1 \forall i = 1, \ldots, m$, and $\hat{\rho}$ is a column vector of functions defined as follows

$$(\hat{\rho}(x))_j = \left( \prod_{i=1, i \neq j}^{m} \rho_i(x_i) \right) \rho_j'(x_j)$$

where $'$ denotes differentiation with respect to the corresponding single variable. $\rho(x) = \prod_{i=1}^{m} \rho_i(x_i)$, $g(x) = (g_1(x)^T, g_2(x)^T, \ldots, g_m(x)^T)^T = Lx$ where $L \in \mathbb{R}^{m \times m}$ is as follows

$$L = \begin{bmatrix} 0 & 0 & \ldots & 0 & \beta_1 \\ \beta_2 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \beta_m & 0 \end{bmatrix}. \quad (21)$$

Then $x = 0 \in \mathbb{R}^n$ is an a.e.u. stable equilibrium point for (14) if and only if the following condition is satisfied:

$$\exists \epsilon > 0 \exists x^T L^T \hat{\rho}(x) + \sum_{i=1}^{m} \prod_{j=1, j \neq i}^{m} \rho_j(x_j) \geq \epsilon \text{ a.e. } x \in X. \quad (22)$$

**4. Examples**

**Example 4.1**

$$\dot{x}_1 = - \sin(x_1)(1 + \cos(x_2)); \dot{x}_2 = - \sin(x_2)(1 + \cos(x_1)) \text{ on } X = [-\pi, \pi] \times [-\pi, \pi]. \quad (23)$$

The system (23) above has $\Lambda = (0, 0)$ as an a.e.u. stable equilibrium point, $(0, \pi), (0, -\pi), (\pi, 0)$, and $(-\pi, 0)$ as saddle points, $(\pi, \pi), (-\pi, -\pi), (\pi, -\pi), (-\pi, \pi)$ as unstable foci. The set $S$ can be thought of as a torus. Equation (23) can be thought of as consisting of two isolated systems $\dot{x}_1 = - \sin(x_1); \dot{x}_2 = - \sin(x_2)$, with an interconnection $g(x_1, x_2) = (- \sin(x_1) \cos(x_2), - \sin(x_2) \cos(x_1))^T$. The two subsystems are individually a.e.u. stable
on \([-\pi, \pi]\) with Lyapunov density (corresponding to initial densities \(\rho_0(x_1) = \chi_{[-\pi, \pi]}(x_1)\), \(\rho_0(x_2) = \chi_{[-\pi, \pi]}(x_2)\), respectively) given by

\[
\rho_1(x_1) = \begin{cases} 
\frac{(\pi - x_1)}{\sin(x_1)} & \text{if } 0 < x_1 < \pi \\
\frac{(-\pi - x_1)}{\sin(x_1)} & \text{if } -\pi < x_1 < 0,
\end{cases}
\]

and

\[
\rho_2(x_2) = \begin{cases} 
\frac{(\pi - x_2)}{\sin(x_2)} & \text{if } 0 < x_2 < \pi \\
\frac{(-\pi - x_2)}{\sin(x_2)} & \text{if } -\pi < x_2 < 0,
\end{cases}
\]

both of which belong to \(L^1([-\pi, \pi])\) for any \(\delta > 0\). Figure 2 is a contour plot of the expression (16) and also a contour plot of \(\rho(x) = \rho_1(x)\rho_2(x)\). This \(\rho\) is the Lyapunov density corresponding to some \(\rho_0(x) \in L^1(\mathbb{A}^c)\). In [8], it was shown that a.e.u. stability with respect to the Lebesgue measure i.e. \((\rho_0(x) = \chi_\mathbb{A}^c(x))\) is equivalent to a.e.u. stability with respect to any finite measure \(m\) with a suitable density \(\rho_0(x)\). We have chosen to scale the plots using natural log and arctangent functions in order to see some features that are not visible while plotting the actual contours. Also, we have voided out a neighborhood of the origin since the density blows up in that region as expected. From the plots, it is clear that Condition (16) is satisfied, and hence the origin is a.e.u. stable for the coupled system (23). In fact, one can actually compute the Lyapunov density directly using \(\rho_0(x) = \chi_\mathbb{A}^c(x)\) (where \(\mathbb{A} = \{0\}\)) using standard FEM techniques. Figure 3 shows the Lyapunov density computed with FEM.

Next, to demonstrate Corollaries (3.2) and (3.3), we consider an example with linear and cascaded coupling.

**Example 4.2** Consider the following system:

\[
\dot{x}_1 = x_1^3 - x_1 + x_2 \quad \dot{x}_2 = -x_2 + x_1 \quad \text{on } X = \left[\begin{array}{c}
-\frac{1}{2} \\
\frac{1}{2}
\end{array}\right] \times \left[\begin{array}{c}
\frac{1}{2} \\
\frac{1}{2}
\end{array}\right].
\]

(24)

On the domain \(X\) specified, \(\Lambda = (0, 0)\) is certainly an equilibrium point. The system (24) can be treated as \(\dot{x}_1 = x_1^3 - x_1; \dot{x}_2 = -x_2\) with an interconnection matrix

\[
L = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix},
\]

(25)

where \(L\) is as in Corollaries (3.2) and (3.3). The isolated systems \(\dot{x}_1 = x_1^3 - x_1; \dot{x}_2 = -x_2\) have an a.e.u. stable equilibrium point at \(x_1 = x_2 = 0\), respectively, and hence their respective Lyapunov densities \(\rho_1\) and \(\rho_2\) are given by:

\[
\rho_1(x_1) = \begin{cases} 
\frac{(x_1 + \frac{1}{2})}{(x_1^3 - x_1)} & \text{if } -\frac{1}{2} < x < 0 \\
\frac{(x_1 - \frac{1}{2})}{(x_1^3 - x_1)} & \text{if } 0 < x < \frac{1}{2},
\end{cases}
\]

(26)

and

\[
\rho_1(x_1) = \begin{cases} 
\frac{(-x_2 + \frac{1}{2})}{x_2} & \text{if } -\frac{1}{2} < x < 0 \\
\frac{(-x_2 - \frac{1}{2})}{x_2} & \text{if } 0 < x < \frac{1}{2},
\end{cases}
\]

(27)
Figure 2. (a) Contour plot of $\ln \left( \nabla x \cdot (g(x) \prod_{i=1}^{2} \rho_i(x_i)) + \sum_{i=1}^{2} \prod_{j=1, j \neq i}^{2} \rho_j(x_j) \right)$ for (23); (b) contour plot of $\arctan(\ln(\rho_1 \rho_2))$ for (23).
Figure 3. (a) Contour plot of $\arctan(\ln(\rho))$ where $\rho$ Lyapunov density corresponding to $\rho_0(x) = \chi_{\Lambda^c}(x)$ for (23) computed using FEM.

Figure 4 shows a plot of the expression in Condition (22) and the purple regions indicate negative values. It is clear that the measure of the set where the expression $x^T L^T \hat{\rho}(x) + \sum_{i=1}^{m} \prod_{j=1, j \neq i}^{m} \rho_j(x_j)$ is less than zero is positive, and hence the system (24) does not have an a.e.u. stable equilibrium point at the origin. This fact is further confirmed in Figure 4 where the vector field plots of the system (24) without and with the linear cascade interconnection (plots (a) and (b), respectively) show that the interconnection actually converts the origin from a stable node to a saddle point (Figure 5).

We finish by showing a modified version of the previous example where Condition (16) is satisfied only almost everywhere, and yet we have a.e.u. stability.

Example 4.3

\[
\dot{x}_1 = x_1^3 - x_1 - x_2^3; \quad \dot{x}_2 = -x_2 \ 	ext{on} \ X = \left[ -\frac{1}{2}, \frac{1}{2} \right] \times \left[ -\frac{1}{2}, \frac{1}{2} \right].
\]  

(28)

Once again, we have that $\Lambda = (0, 0)$ is an equilibrium point. The system (24) can be treated as $\dot{x}_1 = x_1^3 - x_1$; $\dot{x}_2 = -x_2$ along with the interconnection $g(x_1, x_2) = (-x_2^3, 0)$ as in Theorem (3.1). In this case, as seen in Figure 6(a), Condition (16) is satisfied everywhere except on the y-axis where we see a purple color. However, the y-axis is a set of Lebesgue measure zero, and hence (0, 0) is actually a.e.u. stable for the interconnected system (28).
Figure 4. (a) Contour plot of $x^T L^T \hat{\rho}(x) + \sum_{i=1}^{2} \prod_{j=1, j \neq i}^{2} \rho_j(x_j)$ for (24) computed using FEM. Purple indicates regions where the expression is negative.

Figure 5. (a) Vector field plot for $\dot{x}_1 = x_1^3 - x_1; \dot{x}_2 = -x_2$ without interconnection indicating a stable node at the origin; (b) contour plot of (24) with the interconnection indicating a saddle point at the origin.

Figure 6(b) shows the vector field plot of (28) where the origin is clearly a stable node and all the mass is being pulled into the origin.
Figure 6. (a) Contour plot of $\nabla_x \cdot (g(x) \prod_{i=1}^{2} \rho_i(x_i)) + \sum_{i=1}^{2} \prod_{j=1, j \neq i}^{2} \rho_j(x_j)$ indicating that the expression is less than zero on the $y$-axis; (b) vector field plot for (28) with interconnection indicating a stable node at the origin.
Figure 7. (a) Contour plot of $\nabla_x \cdot (g(x) \prod_{i=1}^2 \rho_i(x_i)) + \sum_{i=1}^2 \prod_{j=1, j\neq i} \rho_j(x_j)$ indicating that the expression is less than zero (purple part) on a set of positive Lebesgue measure; (b) vector field plot for $\dot{x}_1 = x_1^3 - x_1 - y_1^3, \dot{x}_2 = -x_2$ on $X = [-\frac{1}{2}, \frac{1}{2}] \times [-2, 2]$ with interconnection indicating a stable focus at the origin. Notice the portions on the top and bottom of the rectangle of positive Lebesgue measure that leave the set $X$. 

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Remark 1 We remark that there could be cases where the origin is a stable node and yet it is not a.e.u. stable because portions of the state space $X$ with positive Lebesgue measure could actually leave the set $X$. Hence, the choice of $X$ also plays an important role here. For example, if we look at $\dot{x}_1 = x_1^3 - x_1 - x_2^3; \dot{x}_2 = -x_2$ on $X = [-\frac{1}{2}, \frac{1}{2}] \times [-2, 2]$, with $f_1(x_1) = -x_1^2; f_2(x_2) = -x_2; g(x_1, x_2) = (-y_1^3, 0)^T$, even though the origin is a stable focus, trajectories starting from a set of positive measure actually leaves $X$ as shown in Figure 7(b). Hence, the origin is not a.e.u. stable and this is confirmed by looking at Figure 7(a) where we see that Condition (16) is not satisfied on a set of positive measure (the portion in purple color).

5. Conclusion
We have proved a necessary and sufficient condition on the kinds of interconnections that preserve the a.e.u. stability of the origin as an equilibrium point for coupled systems assuming that the origin is a.e.u. stable for the isolated systems themselves. The result is akin to a similar Theorem proved in [9] for coupled systems. We conclude by remarking that Theorem (3.1) can be generalized to the case of a general attractor set $\Lambda$ for the coupled system (14) provided the assumptions (1–4) are modified to include some consistency conditions that allow for $\Lambda = \times_{i=1}^{m} \Lambda_i$ to be invariant for (14), where $\Lambda_i \subset X_i$ is an invariant set for the uncoupled system (15).

References